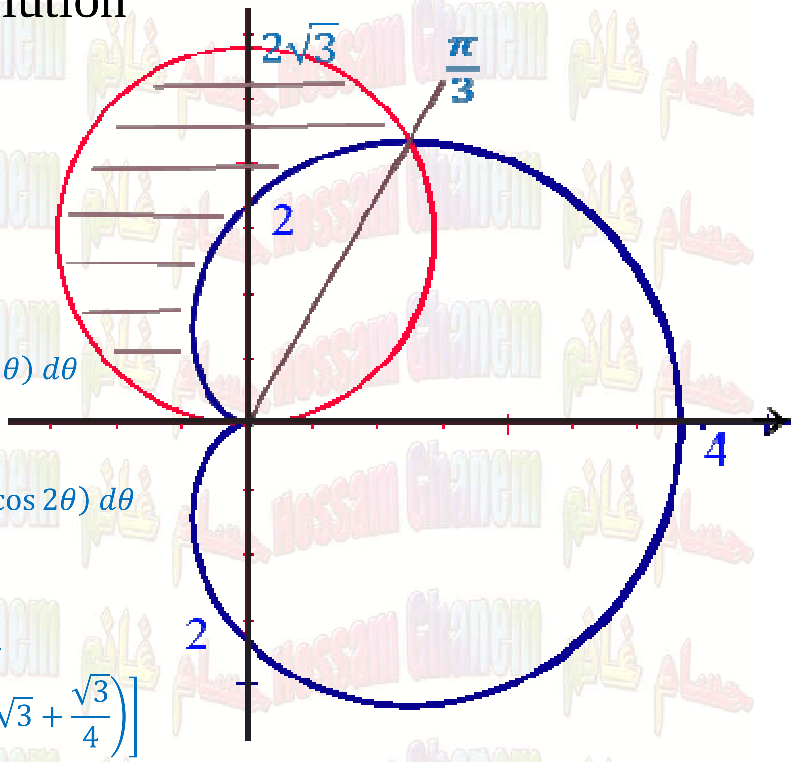
HOSSAM GHANEM

(40) 11.4 Areas In Polar Coordinates

Example 1	Find the area of the region that is outside the graph of $r = 2 + 2\cos\theta$ and inside the graph of $r = 2\sqrt{3}\sin\theta$ (the intersection points of these curves are on the line $\theta = \pi/3$)	17 May 2000

Solution



$$A = \frac{1}{2}\int_a^b r^2\, d\theta$$

$$A = \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} \left(2\sqrt{3}\sin\theta\right)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} (2 + 2\cos\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} 12\sin^2\theta\ \ d\theta - \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} (4 + 4\cos\theta + 4\cos^2\theta)\, d\theta$$

$$= 3\int_{\frac{\pi}{3}}^{\pi} (1 - \cos 2\theta)\, d\theta - \int_{\frac{\pi}{3}}^{\pi} (2 + 2\cos\theta + 1 + \cos 2\theta)\, d\theta$$

$$= 3\left[\theta - \frac{1}{2}\sin\theta\right]_{\frac{\pi}{3}}^{\pi} - \left[3\theta + 2\sin\theta + \frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{3}}^{\pi}$$

$$= 3\left[\pi - 0 - \left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)\right] - \left[3\pi + 0 + 0 - \left(\pi + \sqrt{3} + \frac{\sqrt{3}}{4}\right)\right]$$

$$= 3\pi - \pi + \frac{3\sqrt{3}}{4} - 3\pi + \pi + \sqrt{3} + \frac{\sqrt{3}}{4}$$

$$= \frac{3 + 4 + 1}{4}\sqrt{3} = 2\sqrt{3}$$

Example 2 Find the area enclosed by one loop of the curve $r = \sin 3\theta$ — 27 June 2006

Solution

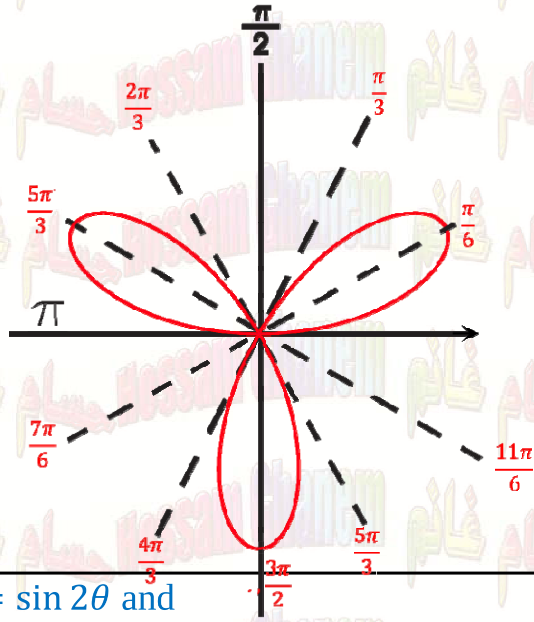
$$A = \frac{1}{2}\int_a^b r^2 d\theta$$

$$= 2 \cdot \frac{1}{2}\int_0^{\frac{\pi}{6}} (\sin 3\theta)^2 d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{2}(1 - \cos 6\theta)\ d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\left[\frac{\pi}{6} - \frac{1}{6}(0)\right] = \frac{\pi}{12}$$



Example 3 Sketch the graph of the equation $r = \sin 2\theta$ and find the area of the region enclosed

Solution

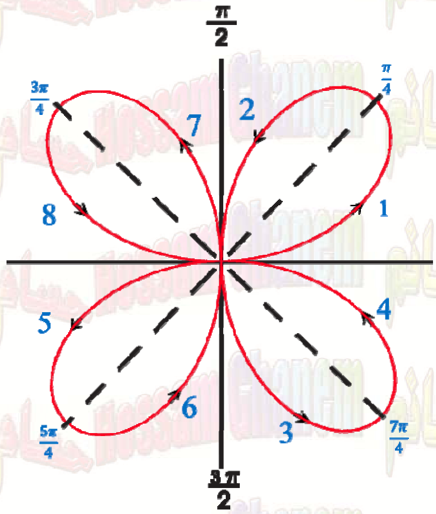
$$A = \frac{1}{2}\int_a^b r^2 d\theta$$

$$= \frac{1}{2} \cdot 8\int_0^{\frac{\pi}{4}} (\sin^2 2\theta) d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \frac{1}{2}(1 - \cos 4\theta)\ d\theta$$

$$= 2\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{2}$$



Example 4 Find the area of the region that is inside the graphs of both polar equations $r = \sin\theta$ and $r = \sin 2\theta$ — 21 January 2004

Solution

intersection points

$$\sin 2\theta = \sin\theta$$

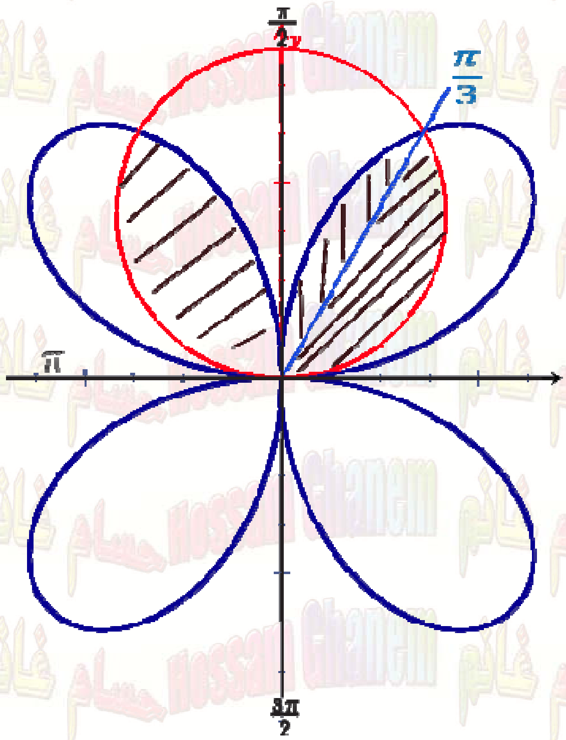
$$2\sin\theta\cos\theta = \sin\theta$$

$$2\sin\theta\cos\theta - \sin\theta = 0$$

$$\sin\theta\,(2\cos\theta - 1) = 0$$

$$\sin\theta = 0 \quad , \quad \cos\theta = \frac{1}{2}$$

$$\theta = 0 \qquad \theta = \frac{\pi}{3}$$



$$A = \frac{1}{2}\int_a^b r^2 d\theta$$

$$I = 2\cdot\frac{1}{2}\int_0^{\frac{\pi}{3}} \sin^2\theta \; d\theta + 2\cdot\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin^2 2\theta \; d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}} (1-\cos 2\theta)\; d\theta + \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1-\cos 4\theta)\; d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{3}} + \frac{1}{2}\left[\theta - \frac{1}{4}\sin 4\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right] + \frac{1}{2}\left[\frac{\pi}{2} - 0 - \left(\frac{\pi}{3} + \frac{\sqrt{3}}{8}\right)\right]$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{8} + \frac{\pi}{4} - \frac{\pi}{6} - \frac{\sqrt{3}}{16} = \frac{\pi}{4} - \frac{3\sqrt{3}}{16}$$

Example 5

40 August 7 , 2011

(2 + 2 + 2 pts) Given the polar curve $r = \frac{\pi}{2} - 2\theta, \quad 0 \le \theta \le \frac{\pi}{2}$

(a) Sketch the curve.

(b) Find the equation of the tangent line to the curve at $\theta = \frac{\pi}{4}$

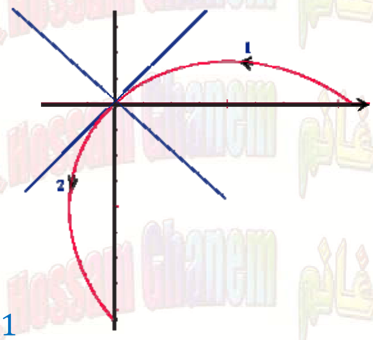
(c) Find the area of the region bounded by the curve and the $x -$ axis

Solution

(a)

	2θ	θ	r
0	0	0	$\frac{\pi}{2}$
1	$\frac{\pi}{2}$	$\frac{\pi}{4}$	0
2	π	$\frac{\pi}{2}$	$-\frac{\pi}{2}$

	θ	r
1	$0 \to \frac{\pi}{4}$	$\frac{\pi}{2} \to 0$
2	$\frac{\pi}{4} \to \frac{\pi}{2}$	$0 \to -\frac{\pi}{2}$



(b)

p

$$r = \frac{\pi}{2} - 2\theta$$

$$r|_{\frac{\pi}{4}} = \frac{\pi}{2} - 2 \cdot \frac{\pi}{4} = 0$$

$$x = r\cos\theta = 0$$

$$y = r\sin\theta = 0$$

$p(0, 0)$

m

$$\frac{dr}{d\theta} = -2 \quad , \quad \sin\theta = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad , \quad \cos\theta = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$m = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{-2 \cdot \frac{1}{\sqrt{2}} + 0}{-2 \cdot \frac{1}{\sqrt{2}} - 0} = 1$$

$$y - y_1 = m(x - x_1)$$

$y = x$

$$A = \frac{1}{2}\int_a^b r^2 d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(\frac{\pi}{2} - 2\theta\right)^2 d\theta$$

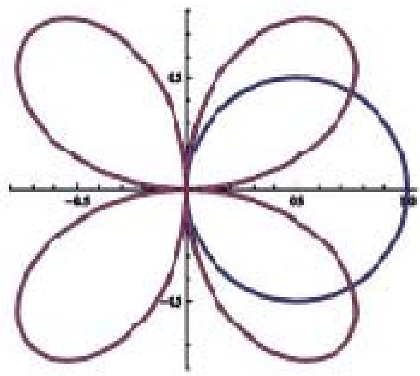
$$= \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{-1}{2}\left[\left(\frac{\pi}{2} - 2\theta\right)^3\right]_0^{\frac{\pi}{4}} = \frac{-1}{12}\left[0 - \left(\frac{\pi}{2}\right)^3\right] = \frac{1}{12} \cdot \left(\frac{\pi}{2}\right)^3 = \frac{\pi^3}{96}$$

Homework

1	Find the area of the region that lies inside both curves $r = \frac{1}{2}$ and $r = \cos\ 2\theta$	26 January 2006
2	Sketch the graphs of the polar equations $r_1 = 1 + \cos\theta$ and $r_2 = 1 - \cos\theta$. Find the area of the region that lies inside both graphs.	18 July 2000
3	Find the area inside the graph of $r = 3\sin\theta$ and outside the graph of $r = 2 - \sin\theta$	12 January 1998

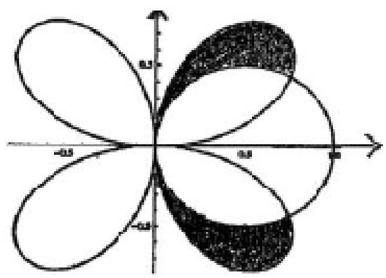
4 32 January 2009 A

The figure below shows the graphs of the polar equations $r = \cos\theta$ and $r = \sin 2\theta$. Find the area of the region that lies inside both curves.



5 35 January 24, 2010

The figure shows the graphs of the polar equations $r = \sin 2\theta$ and $r = \cos\theta$
Find the area of the shaded region.
(4 pts.)



3	Find the area inside the graph of $r = 3\sin\theta$ and outside the graph of $r = 2 - \sin\theta$	12 January 1998

Solution

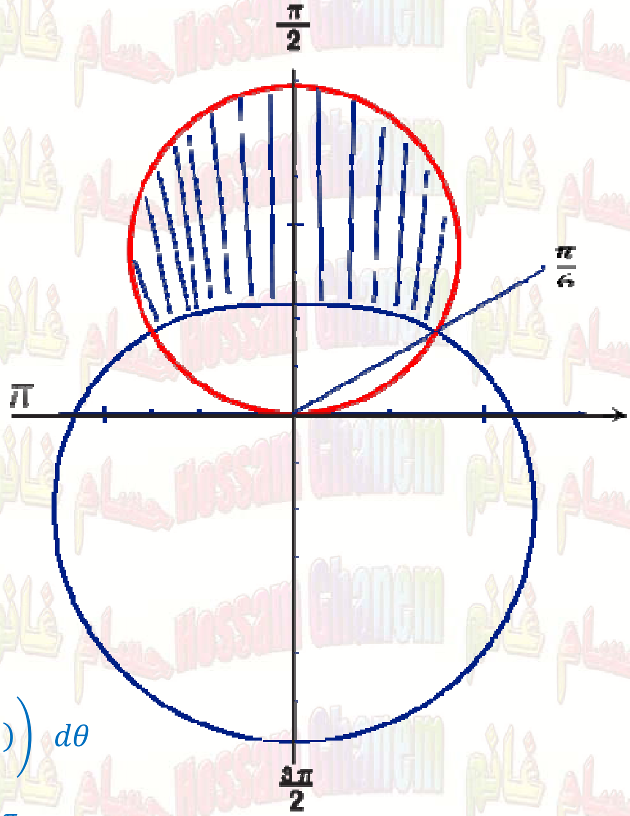
intersection points

$3\sin\theta = 2 - \sin\theta$

$4\sin\theta = 2$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$



$$A = \frac{1}{2}\int_a^b r^2 d\theta$$

$$= \frac{1}{2}\cdot 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3\sin\theta)^2 d\theta - \frac{1}{2}\cdot 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2-\sin\theta)^2 d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (9\sin^2\theta) d\theta - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (4 - 4\sin\theta + \sin^2\theta) d\theta$$

$$= \frac{9}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1-\cos 2\theta)\ d\theta - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(4 - 4\sin\theta + \frac{1}{2}(1-\cos 2\theta)\right)\ d\theta$$

$$= \frac{9}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \left[4\theta + 4\cos\theta + \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{9}{2}\left[\frac{\pi}{2} - 0 - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)\right] - \left[2\pi + 0 + \frac{1}{2}\left(\frac{\pi}{2} - 0\right) - \left(\frac{4\pi}{6} + \frac{4\sqrt{3}}{2} + \frac{1}{2}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)\right)\right]$$

$$= \frac{9\pi}{4} - \frac{9\pi}{12} + \frac{9\sqrt{3}}{8} - 2\pi - \frac{\pi}{4} + \frac{4\pi}{6} + \frac{4\sqrt{3}}{2} + \frac{\pi}{12} - \frac{\sqrt{3}}{8} = \frac{(27 - 9 - 24 - 3 + 8 + 1)\pi}{12} + \frac{(9 + 16 - 1)\sqrt{3}}{8}$$

$$= 0 + \frac{24\sqrt{3}}{8} = 3\sqrt{3}$$